SOME RESULTS ON CRUMPLED CUBES[1]

BY

LLOYD L. LININGER

Introduction. This paper deals with some problems on crumpled cubes. In [6] R. H. Bing raised the following questions: (1) Is the union of a crumpled cube and a 3-cell topologically S^3? (2) Under what conditions will the union of two crumpled cubes sewn together along their boundaries be S^3? (3) Under what conditions will the union of two copies of the same crumpled cube sewn together along their boundaries by the identity homeomorphism be S^3? In [2] Bing proves that the union of two copies of the Solid Alexander Horned Sphere sewn together along their boundaries by the identity homeomorphism is S^3. In [1] B. J. Ball proves that there exists a crumpled cube C such that two copies of C sewn together along their boundaries (not by the identity homeomorphism) is not topologically S^3. In [7] Casler proves that the union of two copies of the Solid Alexander Horned Sphere sewn together along their boundaries is S^3.

In this paper it is proved in §2 that question (1) has an affirmative answer. This result has recently been obtained independently by Norman Hosay [9]. Using this result, it is shown in §3 that if C is a crumpled cube and B is a unit ball in E^3, then there exists an upper semi-continuous decomposition G of B such that (1) B/G is homeomorphic to C, and (2) if g is a nondegenerate element of G, then g is point-like and the intersection of g with the boundary of B is a one-point set. Corollary 2 of §3 shows that if C and D are crumpled cubes, S is a tame 2-sphere in S^3, and C and D are sewn together along their boundaries, then there exists an upper semi-continuous decomposition G of S^3 such that the intersection of each nondegenerate element of G with S is a one-point set and S^3/G is homeomorphic to the union of C and D.

In §4 it is shown that if two copies of a particular "bad" crumpled cube are sewn together by the identity homeomorphism along their boundaries the result is S^3, and it is shown that if two copies of a particular "nice" crumpled cube are sewn together by the identity homeomorphism along their boundaries the result is not S^3.

1. Notation and terminology. If A is a set in a topological space, then Cl A and Bd A denote the closure and boundary, respectively, of A. If S is a 2-sphere in

E^3, then Int S and Ext S denote the interior and exterior, respectively, of S. The usual metric for E^3 is d.

A set C is a crumpled cube if and only if C is homeomorphic to the union of a 2-sphere and its interior in E^3.

A space X is a disk-with-holes if and only if X is a disk or there exist disjoint disks $D_1, \cdots, D_n$ in the interior of a disk D such that X is homeomorphic with $D - \bigcup_{i=1}^{n} \operatorname{Int} D_i$.

For terms not defined in this paper the reader may see [4] or [6].

2. The sum of a crumpled cube and a 3-cell. In this section it is proved that the sum of a crumpled cube and a 3-cell is topologically S^3. Theorem 1 is used to obtain this result. The proof of Theorem 1 given here relies heavily on the Side Approximation Theorem proved by Bing. For completeness this theorem is stated below.

BING'S SIDE APPROXIMATION THEOREM. *For each 2-sphere S in E^3, each positive number e, and each component U of $E^3 - S$, there esists a homeomorphism h from S onto a polyhedral 2-sphere such that (1) if x is in S, then $d(x, h(x)) < e$, (2) there exists a finite collection of disjoint disks $D_1, \cdots,$ and D_n on $h[S]$, each of diameter less than e such that $h[S] - \bigcup_{i=1}^{n} D_i \subset U$, and (3) there exists a finite collection of disjoint disks $E_1, \cdots,$ and E_m on S, each of diameter less than e such that $S - \bigcup_{i=1}^{m} E_i$ is contained in $\operatorname{Int} h[S]$ if U is the exterior of S or is contained in $\operatorname{Ext} h[S]$ if U is the interior of S* [4].

THEOREM 1. *If C is a crumpled cube in E^3 with boundary S, A is a compact subset of $\operatorname{Int} C$, and e is a positive number, then there exist a homeomorphism h' from C into E^3 and a polyhedral 2-sphere S' such that S' is homeomorphically within e of S, $h'[C] \subset \operatorname{Int} S'$, if $x \in C$ then $d(x, h'(x)) < e$, and if $x \in A$ then $h'(x) = x$.*

Proof. There exists a positive number e' such that if T is a 2-sphere within e' of S, then $A \subset \operatorname{Int} T$. Let e'' denote $(1/11) \min \{e, e', (1/10) \operatorname{diam} C\}$. Since S is a 2-sphere, then by Bing's Side Approximation Theorem, there exists a polyhedral 2-sphere S_1 such that (a) S_1 is homeomorphically within e'' of S, (b) there exists a finite number of disjoint disks $E_1, \cdots,$ and E_m on S_1, each of diameter less than e'', such that $\operatorname{Cl}(S_1 - \bigcup_{i=1}^{m} E_i)$ is contained in $\operatorname{Ext} S$, and (c) there exists a finite number of disjoint disks $D_1, \cdots,$ and D_n on S, each of diameter less than e'', such that $\operatorname{Cl}(S - \bigcup_{i=1}^{n} D_i)$ is contained in $\operatorname{Int} S_1$. For each i, $i = 1, 2, \cdots, m$, let K_i denote a polyhedral disk such that (a) $\operatorname{Bd} K_i = \operatorname{Bd} E_i$, (b) $K_i - \operatorname{Bd} K_i$ is contained in $\operatorname{Ext} S_1$, (c) $(S_1 - \bigcup_{i=1}^{m} E_i) \cup (\bigcup_{i=1}^{m} K_i)$ is a 2-sphere S', and (d) S' is homeomorphically within e'' of S.

It will be shown that there exist a polyhedral 3-manifold-with-boundary M and a homeomorphism h with the following properties: (1) $(S \cup \operatorname{Int} S) \subset \operatorname{Int} M$, (2) for each i, $i = 1, 2, \cdots, m$, no simple closed curve in M links $\operatorname{Bd} E_i$, (3) if D_i' is a polyhedral disk such that $\operatorname{Bd} E_i = \operatorname{Bd} D_i'$ then there does not exist a 2-sphere in M that contains D_i' in its interior, (4) $M - S'$ has one and only one component

V with diameter more than one half the diameter of C, (5) S', S_1, and M are in relative general position, (6) if W is a component of $M \cap S'$ such that W does not separate M, and W_1, W_2, $\cdots$, and W_k are components of $M \cap S'$ such that (i) $\bigcup_{i=1}^{k} W_i$ does not separate M, (ii) $\bigcup_{i=1}^{k} W_i \cup W$ does separate M, and (iii) $W \cup \bigcup_{i=1}^{k} W_i$ is contained in some component of $\mathrm{Cl}(M - V)$, then there exists a simple closed curve J in M such that J intersects $\bigcup_{i=1}^{k} W_i$ in a one-point set, J intersects W in a one-point set, and the diameter of each component of $J \cap V$ is less than e'', (7) if x is in $V \cap \mathrm{Int}\, S_1$, then $h(x) = x$, (8) h is from M into $\mathrm{Int}\, S'$, and (9) for each x in M, $d(x, h(x))$ is less than $11e''$. This will complete the proof of Theorem 1 since if h' denotes the restriction of h to C then h' and S' satisfy the conclusion of Theorem 1.

There exists a positive number e_1 such that if T is a triangulation of E^3 with mesh e_1, N is the union of all 3-simplexes of T which intersect C, and M is the union of all 3-simplexes is the 2nd barycentric subdivision of T which intersect N, then M will be a polyhedral 3-manifold-with-boundary which satisfies properties (1)–(6).

The proof that there exists a homeomorphism h from M into $\mathrm{Int}\, S'$ that satisfies conditions (7), (8) and (9) is broken into three parts. Let $F_1, \cdots$, and F_p be the components of $M - V$. In Part I it is proved that if for some j, there exists a positive integer i less than or equal to p such that $(\mathrm{Cl}\, V) \cap F_i$ is contained in K_j, then there exists a homeomorphism h_i from M into E^3 such that $h_i[F_i] \subset \mathrm{Int}\, S'$, $h_i[M]$ satisfies conditions (1)–(6), if V' is the large component of $h_i[M] - S'$, then $h_i[M] - V'$ has at most $p - 1$ components, and if x is in $V \cap \mathrm{Int}\, S_1$ then $h_i(x) = x$. In Part II the following is proved: If for each j such that $j \leq p$, there does not exist an integer i such that $F_i \cap \mathrm{Cl}\, V$ is contained in K_j, then there exists an integer k, $1 \leq k \leq m$, and a homeomorphism h_1 from M into E^3 such that $h_1[M]$ satisfies conditions (1)–(7), if V'' denotes the large component of $h_1[M] - S'$, then $h_1[M] - V''$ has at most p components, and $h_1[F_1] \cap \mathrm{Cl}\, V''$ is contained in K_k. In Part III, by using induction and Parts I and II it is shown that there exists a homeomorphism h with the desired properties.

PROOF OF PART I. In this part, it is supposed that for some j, there exists an integer i such that $(\mathrm{Cl}\, V) \cap F_i$ is contained in K_j. A homeomorphism h_i from M into E^3 will be constructed such that $h_i[F_i] \subset \mathrm{Int}\, S_2'$, $h_i[M]$ satisfies conditions (1)–(7), and if V' denotes the large component of $h_i[M] - S'$ then $h_i[M] - V'$ has at most $p - 1$ components.

Let $X_1^1, \cdots, X_{t(1)}^1$ be the distinct components of $F_i - K_j$ such that for each u, $(\mathrm{Cl}\, X_u^1) \cap \mathrm{Cl}\, V \neq \varnothing$, and if W is a component of $F_i - K_j$ such that $(\mathrm{Cl}\, W) \cap \mathrm{Cl}\, V \neq \varnothing$, then there exists a z such that $W = X_z^1$. Let $X_1^2, \cdots$, and $X_{t(2)}^2$ be the distinct components of $F_i - \mathrm{Cl}(K_j \cup [\bigcup_{k=1}^{t(1)} X_k^1])$ such that for each u less than or equal to $t(2)$, $(\mathrm{Cl}\, X_u^2) \cap \mathrm{Cl}(V \cup \bigcup_{k=1}^{t(1)} X_k^1) \neq \varnothing$. Since $F_i - K_j$ has only a finite number of components, there exists an integer r such that if the above process is continued, for each u and w, $X_u^w \neq \varnothing$ and $F_i = \mathrm{Cl}(\bigcup_{w=1}^{r} \bigcup_{u=1}^{t(w)} X_u^w)$.

For each u and w such that w is greater than 1, $(\operatorname{Cl} X_u^w) \cap \operatorname{Cl}(\bigcup_{k-1}^{t(w-1)} X_k^{w-1})$ is a collection of disk-with-holes. Let $Y_{wu1}, \cdots$, and $Y_{wuq(w,u)}$ denote the components of $(\operatorname{Cl} X_u^w) \cap (\operatorname{Cl}[\bigcup_{j=1}^{t(w-1)} X_j^{w-1}])$ and let Y_{wu} denote $\bigcup_{j=1}^{q(w,u)} Y_{wuj}$. For each w, u and z, let T_{wuz} be a disk in K_j' such that (1) $Y_{wuz} \subset T_{wuz}$, and (2) $\operatorname{Bd} T_{wuz} \subset \operatorname{Bd} Y_{wuz}$. Let T_{wuz}' be a slight thickening of T_{wuz} with the following properties: (1) T_{wuz}' is polyhedral and topologically equivalent to $T_{wuz} \times [0,1]$, (2) T_{wuz}' is disjoint from the 0-skeleton of M, and (3) If $T_{wuv} \subset T_{wuz}$ then $T_{wuv} \subset \operatorname{Int} T_{wuz}$. Let T_{wu}' denote $\bigcup_{z=1}^{q(w,u)} T_{wuz}'$, let T_{wuz}'' denote the component of $T_{wuz}' - K_j$ that intersects $\bigcup_{u=1}^{t(w-1)} X_u^{w-1}$, let T_{wu}'' denote $\bigcup_{z=1}^{q(w,u)} T_{wuz}''$, and let T_w'' denote $\bigcup_{u=1}^{t(w)} T_{wu}''$. The remainder of the proof of Part I is now broken into two steps.

Step I. In Step I the following is proved: There exists a homeomorphism f_1 from M into E^3 with the following properties: (1) $f_1[M]$ has properties (1)–(7), (2) If V' is the large component of $f_1[M] - S'$, then $f_1[M] - V'$ has at most p components, (3) $f_1[F_1]$ is contained in $(\operatorname{Int} S') \cup (F_1 - \operatorname{Cl} X_1^w) \cup T_{w1}''$, and (4) If $k \neq 1$, then either $(f_1[\operatorname{Cl} X_k^w]) \cap K_j = f_1[(\operatorname{Cl} X_k^w) \cap K_j]$ or $(f_1[\operatorname{Cl} X_k^w]) \cap K_j = \varnothing$.

PROOF OF STEP I. *Case* 1. $Y_{w1} = Y_{w11}$. Let Y_{w1}^* denote the disk on the boundary of T_{w1}'' that is disjoint from K_j and corresponds to Y_{w1}. See Figure 1. A homeo-
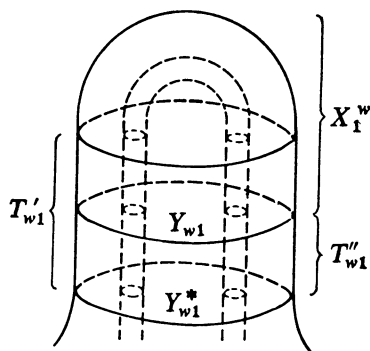

FIGURE 1

morphism f_1 is constructed as follows: Let $W_1, \cdots$, and $W_{g(1)}$ be the components of $M - \operatorname{Bd} T_{w1}''$, and assume $\operatorname{Int} S_1 \cap V$ is contained in W_1 and $X_1^w = W_2$. If x is in W_1 then $f_1(x) = x$. If $(\operatorname{Cl} W_1) \cup \operatorname{Int} T_{w1}'' = \varnothing$ and W_1 is contained in the unbounded component of $E^3 - (T_{w1}'' \cup (\operatorname{Cl} X_1^w))$, then $f_1(x) = x$. If $W = \{W_i : W_i \cap \operatorname{Int} T_{w1}'' = \varnothing$ and either $W_i = W_2$ or W_i is not contained in the unbounded component of $E^3 - (T_{w1}'' \cup (\operatorname{Cl} X_1^w))\}$ and $W' = \bigcup\{\operatorname{Cl} W_i$: there exists an integer s such that $W_s \in W$ and $\operatorname{Cl} W_s \cup \operatorname{Cl} W_i \neq \varnothing\}$, then $f_1[W']$ is contained in T_{w1}'' and f_1 leaves W' pointwise fixed on $\operatorname{Bd} T_{w1}''$. Each of the remaining W_k's (if there are any) is contained in T_{w1}'' and the homeomorphism f_1 can be extended to them as shown in Figure 2.

Case 2. $Y_{w1} = Y_{w11} \cup \cdots \cup Y_{w1q(w,1)}$. There exists a simple closed curve J on $\operatorname{Bd} M$ such that (1) $J \subset \operatorname{Cl}(V \cup \bigcup_{k=1}^{w-1} \bigcup_{u=1}^{t(k)} X_u^k)$, (2) The diameter of each compo-
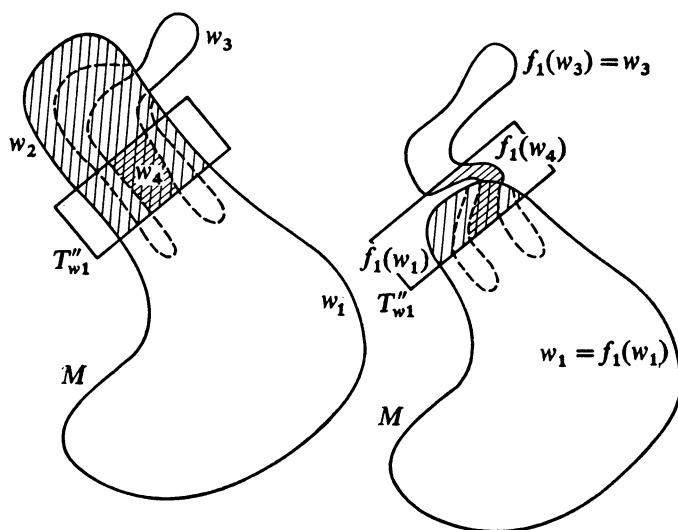
FIGURE 2

nent of $J \cup V$ is less than e, and (3) $J \cap Y_{w11}$ is a one-point set, and it can be assumed that $J \cap Y_{w12}$ is a one-point set while J is disjoint from each of $Y_{w13}, \cdots$, and $Y_{w1q(w,1)}$. Shrink F_1 slightly so that $J \cap F_1$ misses the modified F_1 and if J intersects V, pull J off V slightly. See Figure 3. Then there exists a homeomor-
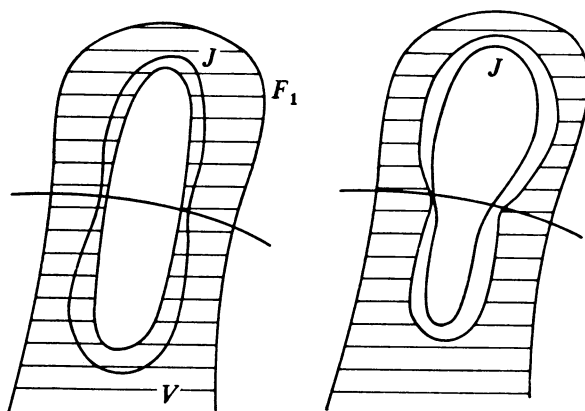

FIGURE 3

phism f^* from the modified M into E^3 with the following properties: (1) If x is in $M - T_{w12}$ then $f^*(x) = x$, (2) f^* maps $T_{w12} \cap$ [modified M] as shown in Figure 4. Note that $J \cap (E^3 - T_{w12})$ is the center of $f^*[T_{w12} \cap$ modified $M]$ $\cap (E^3 - T_{w12})$, and (3) f^*[modified M] satisfies conditions (1)–(7), and if V^* is the large component of f^*[modified M] $- S'$, then f^*[modified M] $- V^*$ has at most p components. After $q(w,1) - 1$ repetitions of this argument, the modified Y_{w1} is contained in Y_{w11}. See Figure 5. An argument like the one given
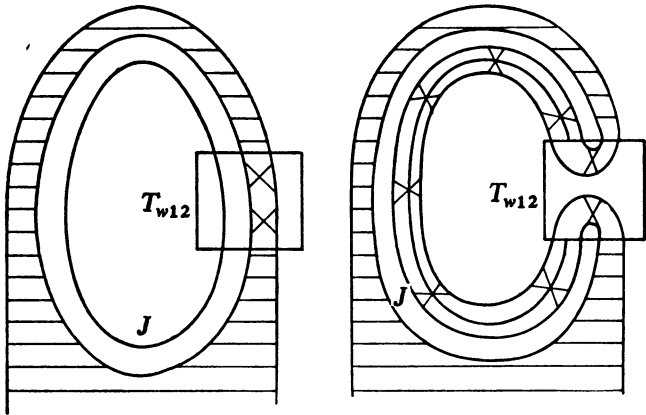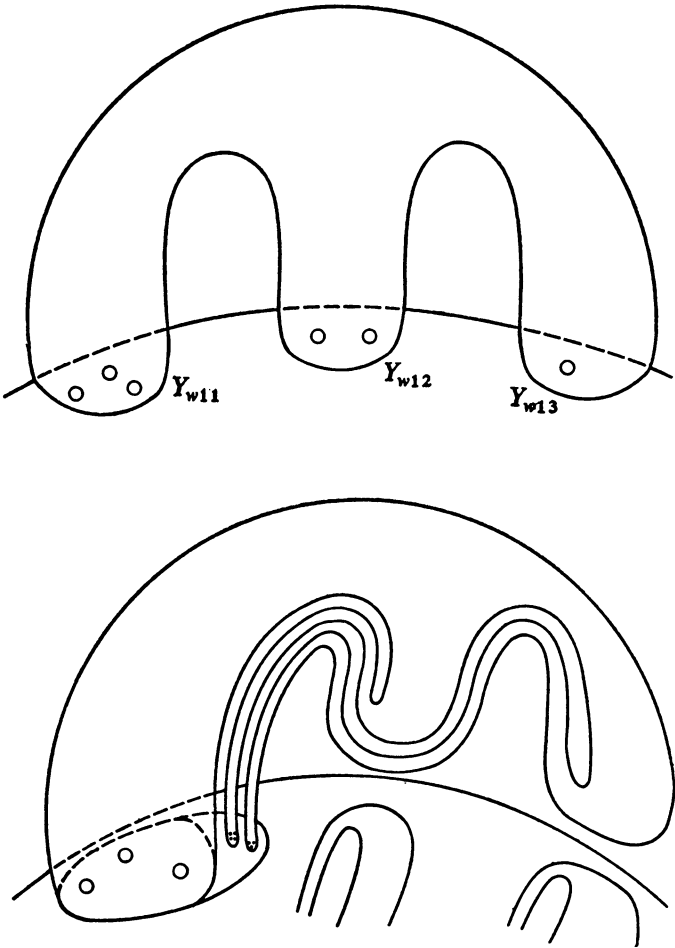
FIGURE　4

FIGURE 5

in Case 1 using T''_{w11} in place of T''_{w1} gives the existence of a homeomorphism f_1 with the desired properties. This concludes the proof of Step I.

It follows that there exist homeomorphisms $f_2, \cdots,$ and $f_{t(w)}$ satisfying the following: (1) $f_{t(w)} \cdots f_2 f_1$ is a homeomorphism from M into E^3, (2) $f_{t(w)} \cdots f_1 [M]$ satisfies conditions (1)–(7) (3) If V'' is the large component of $f_{t(w)} \cdots f_1 [M] - S'$ then $f_{t(w)} \cdots f_1 [M] - V''$ has at most p components, and (4) $f_{t(w)} \cdots f_1 [M]$ is contained in

$$(M - \text{Cl}[\bigcup_{k=1}^{t(w)} X_k^w]) \cup \bigcup_{k=1}^{t(w)} T''_{wk}.$$

Step II. In Step II it is proved that there exists a homeomorphism h_i from M into E^3 such that $h_i[F_i] \subset \text{Int} S'$, $h_i[M]$ satisfies conditions (1)–(7), and if V' is the large component of $h_i[M] - S'$ then $K_i[M] - V'$ has at most $p-1$ components. This will conclude the proof of Part I.

PROOF OF STEP II. Using an argument similar to the one given in Step I, it can be shown that there exists a homeomorphism $f_{t(w)+1}$ from $f_{t(w)} \cdots f_1[M]$ into E^3 with the following properties: (1) $f_{t(w)+1} \cdots f_1[M]$ satisfies conditions (1)–(7), (2) Each component of $f_{t(w)} \cdots f_1[M] - K_j$ that intersects X_1^{w-1} is mapped into $(M - [\bigcup_{k=1}^{t(w)} X_k^w \cup X_1^{w-1}]) \cup [\bigcup_{k=1}^{t(w-1)} T''_{(w-1)k} \cup T''_w]$, and (3) If V''' is the large component of $f_{t(w)+1} \cdots f_1[M] - S'$ then $f_{t(w)+1} \cdots f_1[M] - V'''$ has at most p components. After $t(1) + t(2) + \cdots + t(w-1) - 1$ additional repetitions of this argument the homeomorphism h_i is obtained. Since $h_i[F_i]$ is contained in the union of T''_1 and a small neighborhood of V, it follows that $h_i[F_i]$ is contained in Int S'.

PART II. In this part, it is supposed that for each integer j less than or equal to p, there does not exist an integer i such that $F_i \cap \text{Cl} V$ is contained in K_j. It is shown that there exist an integer k, $1 \leq k \leq m$, and a homeomorphism h_1 from M into E^3 such that (1) $h_1[M]$ satisfies conditions (1)–(7), (2) If V'' denotes the large component of $h_1[M] - S'$ then $h_1[M] - V''$ has at most p components, and (3) $h_1[F_1] \cup \text{Cl} V''$ is contained in K_k.

PROOF OF PART II. An argument similar to the one given in the proof of Part I, Step I, Case 2 shows that there exists a homeomorphism h_1 from M into E^3 such that h_1 has the desired properties

PART III. In this part, it is shown that there is a homeomorphism h which satisfies the conclusion of Theorem 1.

PROOF OF PART III. Assume the F_i's are indexed so that there exists an integer u, $1 \leq u \leq p$, with the property that if $i < u$ then there exists an integer k, $1 \leq k \leq m$, such that $F_i \cap \text{Cl} V$ is contained in K_k and if $i \geq u$ then there does not exist an integer k, $1 \leq k \leq m$, such that $F_i \cap \text{Cl} V \subset K_k$. If u is greater than 1, apply Part I to F_1 and obtain a homeomorphism h_1 from M into E^3 that maps F_1 into Int S'. Then apply Part I to $h_1[F_2]$ obtaining h_2. Continue this process to obtain homeomorphisms $h_3, \cdots,$ and h_{u-1}. Then by applying Part II to $h_{u-1} \cdots h_1[F_u]$, a homeo-

morphism h_u is obtained, and the results of Part I are then applied to $h_u \cdots h_1[F_u]$ to obtain h_{u+1}. Then by using Part I or Part II, depending upon $h_{u+1} \cdots h_1[F_{u+1}]$, a homeomorphism h_{u+2} is obtained. Continue this process. Let h denote the composition of the h_i's. Clearly h satisfies conditions (1)–(8). Condition (9) follows since for each i less than or equal to p, the diameter of F_i is less than $5e''$, the diameter of $h[F_i]$ is less than $5e''$, and h moves no point of V more than e''. This concludes the proof of Theorem 1.

THEOREM 2. *If C is a crumpled cube in S^3 and e is a positive number, then there exists a homeomorphism h from C into S^3 such that the closure of $S^3 - h[C]$ is a 3-cell and if x belongs to C then $d(x, h(x))$ is less than e.*

Proof. Let f be a positive sequence such that the series whose sequence of terms is f converges to a number less than e, and let $C_1, C_2, \cdots$ be a sequence of closed sets such that for each i, $C_i \subset \text{Int } C_{i+1}$, and $\bigcup_{i=1}^{\infty} C_i = \text{Int } C$. Let S denote the boundary of C.

By Theorem 1 there exists a polyhedral 2-sphere S_1 and a homeomorphism h_1 from C into S^3 such that S_1 is homeomorphically within $f(1)$ of S, $h_1[C] \subset \text{Int } S_1$ if $x \in C$ then $d(x, h_1(x)) < f(1)$, and if $x \in C_1$ then $h_1(x) = x$.

Let e_2 denote $(1/3) \min \{f(2), d(S_1, h_1[S])\}$. Then by Theorem 1 there exists a polyhedral 2-sphere S_2 and a homeomorphism h'_2 from $h_1[C]$ into S^3 such that S_2 is homeomorphically within e_2 of $h_1[S]$, $h'_2 h_1[C] \subset \text{Int } S_2$, if $x \in h_1[C_2]$ then $h'_2(x) = x$, and if $x \in h_1[C]$ then $d(x, h'_2(x)) < e_2$. Let h_2 be $h'_2 h_1$. It follows that h_2 maps C Int S_2, if $x \in C$ then $d(x, h_2(x)) < f(1) + f(2)$, and $S_2 \subset \text{Int } S_1$.

Proceeding by induction, there exists a 2-sphere S_n and a homeomorphism h'_n from $h_{n-1}[C]$ into S^3 such that S_n is homeomorphically within $f(n)$ of $h_{n-1}[S]$, $h'_n h_{n-1}[C] \subset \text{Int } S_n$, if $x \in h_{n-1}[C_n]$ then $h'_n(x) = x$, $S_n \subset \text{Int } S_{n-1}$, and if $x \in h_{n-1}[C]$ then $d(x, h'_n(x)) < f(n)$. Let h_n be $h'_n h_{n-1}$.

The sequence $h_1, h_2, \cdots$ is uniformly convergent. Let h be the limit of $h_1, h_2, \cdots$. It follows that h is uniformly continuous and into S^3.

It is to be proved that h is one-to-one.

LEMMA 1. *The restriction of h to the interior of C is one-to-one.*

Proof. Suppose x and y are distinct points in Int C. There exists a positive integer j such that x and y are in C_j. Hence for each integer $k \geq j$, $h_k(x) = h_j(x)$, $h_k(y) = h_j(y)$, and $h_j(x) \neq h_j(y)$. Then $h(x) \neq h(y)$.

LEMMA 2. *For each x in $h[\text{Int } C]$, $\lim h_i^{-1}(x)$ exists and is equal to $h^{-1}(x)$.*

Proof. $\text{Sup} \{d(h_i^{-1}(x), h_{i+1}^{-1}(x)): x \text{ in } h[C]\} \leq f(i)$. Hence the sequence $h_1^{-1}, h_2^{-1}, \cdots$ is uniformly convergent on $h[C]$ and therefore, for each x in $h[C]$, $\lim h_i^{-1}(x)$ exists. Suppose x belongs to $h[\text{Int } C]$. By Lemma 1 there is one and only one point y in Int C such that $h(y) = x$ or $y = h^{-1}(x)$. There exists a positive integer j such that $y \in C_j$, hence $h(y) = h_j(y)$ and $\lim h_i^{-1}(x) = h_j^{-1}(x) = y$.

LEMMA 3. *The function h is a homeomorphism from C.*

Proof. It suffices to show h is one-to-one from C. Suppose h is not one-to-one. Then there exist distinct points x and y in C such that $h(x) = h(y)$. Let $x_1, x_2, \cdots$, and $y_1, y_2, \cdots$ be sequences such that for each i, x_i and y_i belong to Int C, the sequence $\{x_i\}$ converges to x, and the sequence $\{y_i\}$ converges to y. Then the sequence $\{h(x_i)\}$ converges to $h(x)$ and the sequence $\{h(y_i)\}$ converges to $h(y)$ which is equal to $h(x)$. Let e' be a positive number less than $d(x, y)/2$. Since h is uniformly continuous on $h[\text{Int} C]$, there is a positive number r such that if $d(p, q) < r$, then $d(h^{-1}(p), h^{-1}(q)) < e'$. But there exists a positive integer i such that $d(h(x_i), h(y_i)) < r$ and $d(x_i, y_i) > e'$. This is a contradiction.

LEMMA 4. *If x belongs to C then $d(x, h(x))$ is less than e.*

Proof. $d(x, h(x)) \leqq \lim_{n \to \infty} (d(x, h_1(x)) + d(h_1(x), h_2(x)) + \cdots + d(h_{n-1}(x), h_n(x)))$
$\leqq \sum_{i=1}^{\infty} f(i) < e$.

LEMMA 5. *The closure of $S^3 - h[C]$ is a 3-cell.*

Proof. For each positive integer i, S_i is homeomorphically within $f(i)$ of $h_{i-1}[S]$ and $h_{i-1}[S]$ is homeomorphically within $\sum_{j=i}^{\infty} f(j)$ of $h[S]$. Therefore S_i is homeomorphically within $f(i) + \sum_{j=i}^{\infty} f(j)$ of $h[S]$. For each positive integer i, S_i is a polyhedral 2-sphere in Ext $h[C]$. It follows from Theorem 2.1 of [5] that the closure of $S^3 - h[C]$ is a 3-cell. This concludes the proof of Theorem 2.

3. **Crumpled cubes as decomposition spaces.** The following definitions and terminology will be used in §3. If r is a positive number, $B(r)$ will denote the ball in E^3 with center at the origin and radius r. If S and T are 2-spheres in E^3 and $S \subset \text{Int} T$, then $[S, T]$ will denote Cl(Int $T - \text{Int} S$). Suppose r and s are positive numbers and $r < s$. A cellular subdivision $\{E_1, \cdots, E_q\}$ of $[\text{Bd } B(r), \text{Bd } B(s)]$ is a canonical subdivision of $[\text{Bd} B(r), \text{Bd} B(s)]$ if and only if there exist disks $D_1, \cdots,$ and D_q on Bd $B(r)$ such that (1) for each i, there exists a j such that if p is in E_i, then there exists a point t in D_j and a real number c such that $p = ct$, (2) if $i \neq j$ and $D_i \cap D_j$ is nonempty, then $D_i \cap D_j \subset \text{Bd} D_i \cap \text{Bd} D_j$, (3) for each i, $E_i \cap B(r)$ is nonempty, and (4) for each i and j, the diameter of D_i is at most twice the diameter of D_j. Suppose S_1 and S_2 are tame 2-spheres in E^3 and S_2 is contained in Int S_1. A cellular subdivision $\{E_1, \cdots, E_q\}$ of $[S_2, S_1]$ is an A-subdivision of $[S_2, S_1]$ if and only if there exists a homeomorphism h from $[S_2, S_1]$ onto $[\text{Bd} B(1), \text{Bd} B(2)]$ such that $\{h[E_1], \cdots, h[E_q]\}$ is a canonical subdivision of $[\text{Bd} B(1), \text{Bd} B(2)]$. If X is a topological space and G is an upper semi-continuous decomposition of X, then X / G will denote the decomposition space associated with G.

THEOREM 3. *If C is a crumpled cube and $\text{Cl}(S^3 - C)$ is a 3-cell, then there exists an upper semi-continuous decomposition G of $B(1)$ such that the intersection*

of each nondegenerate element of G with the boundary of $B(1)$ is a one-point set, each nondegenerate element of G is point-like, and $B(1)/G$ is homeomorphic to C.

Proof. Let S denote the boundary of C, and let e_1 denote $\min\{1, 1/10$ diameter of $C\}$. There exists a polyhedral 2-sphere S_1 in Ext S such that S_1 is homeomorphically within e_1 of S. Now let e_{11} be $(1/2)d(C, S_1)$. By the Side Approximation Theorem there exists a polyhedral 2-sphere S_{11} with the following properties: (1) S_{11} is homeomorphically within e_{11} of S, (2) there exist disjoint disks $D_{11}, \cdots,$ and $D_{1n(1)}$ on S, each of diameter less than e_{11}, such that $\mathrm{Cl}(S - \bigcup_{i=1}^{n(1)} D_{1i})$ is contained in Ext S_{11}, and (3) there exist disjoint disks $K_{11}, \cdots,$ and $K_{1m(1)}$ on S_{11}, each of diameter less than e_{11}, such that $\mathrm{Cl}(S_{11} - \bigcup_{i=1}^{m(1)} K_{1i})$ is contained in Int S. Let e_{12} denote

$$(1/3)\min\{\min\{d(D_{1i}, D_{1j}): i \neq j, 1 \leq i \leq n(1), \text{ and } i \leq j \leq n \ (1)\},$$

$$\min\{d(K_{1i}, K_{1j}): i \neq j, 1 \leq i \leq m(1), \text{ and } 1 \leq j \leq m(1)\},$$

$$\min\{d(D_{1i}, S_{11} - \bigcup_{j=1}^{m(1)} K_{1j}): 1 \leq i \leq n(1)\}\}.$$

Let T_1 be a triangulation of S^3 with mesh less than e_{12}, let N be the union of all 3-simplexes of T which intersect $[S, S_1]$, and let M_{11} be the union of all 3-simplexes in the 2nd barycentric subdivision of T which intersect N. Then M_{11} is a polyhedral 3-manifold with boundary and it can be assumed that M_{11} and S_{11} are in relative general position.

Let M_1 denote the union of M_{11} and $[S_{11}, S_1]$. Now $M_1 - [S_{11}, S_1]$ has a finite number of components, and let $F_{11}, \cdots,$ and $F_{1r(1)}$ be the components of $\mathrm{Cl}(M_1 - [S_{11}, S_1])$. It follows from the Side Approximation Theorem that for each i, F_{1i} has diameter less than $4e_1 + 2e_{12}$ which is less than $5e_1$.

There exists an A-subdivision $\{E_{11}, \cdots, E_{1q(1)}\}$ of $[S_{11}, S_1]$ such that (1) for each i, $i \leq n(1)$, there exists a positive integer j such that Bd $D_{1i} \subset$ Int E_{1j}, and (2) for each i, $i \leq q(1)$, the diameter of E_{1i} is less than $5e_1$. Let A_{1i}^* and A_{1i}, $i \leq q(1)$, be defined as follows: $A_{1i}^* = \bigcup\{X: X \in \{E_{11}, \cdots, E_{1q(1)}\}$ or $X \in \{F_{11}, \cdots, F_{1r(1)}\}$ and $d(X, E_{1i}) < e_1\}$ and A_{1i} is the component of A_{1i}^* that contains E_{1i}. Note that for each p in S, there exists an integer j such that p is in A_{1j}. For each p in S, denote $\bigcup\{A_{1i}: p \in A_{1i}\}$ by A_{p1}. Now for each i, the diameter, of A_{1i} is at most $17e_1$, hence for each p, the diameter of A_{p1} is at most $34e_1$.

Then there exists a homeomorphism h_1 from $S_1 \cup \mathrm{Int}\, S_1$ onto $B(1.1)$ with the following properties: (1) $h_1[[S_{11}, S_1]] = [\mathrm{Bd}\, B(.9), \mathrm{Bd}\, B(1.1)]$, (2) $h_1[C]$ is contained in Int $B(1.01)$, and (3) $\{h_1[E_{11}], \cdots, h_1[E_{1q(1)}]\}$ is a canonical subdivision of $[\mathrm{Bd}\, B(.9), \mathrm{Bd}\, B(1.1)]$ into $q(1)$ cells.

Now M_1 is a 3-manifold-with-boundary and S is a compact subset in Int M_1. It follows that there exists a positive number e_{13} such that, if Q is a set with diameter less than $34e_{13}$ and the intersection of Q with S exists, then there exists a cell R such that $Q \subset \mathrm{Int}\, R \subset R \subset \mathrm{Int}\, M_1$.

Let e_2 denote $\min\{d(S, \mathrm{Bd}\, M_1), e_{13}, 1/4\}$. There exists a polyhedral 2-sphere S_2 in Ext S such that S_2 is homeomorphically within e_2 of S. Let e_{21} be $(1/3)\min\{d(S, S_2), \min\{d(\mathrm{Bd}\, D_{1i}, \mathrm{Bd}\, E_{1j}): \mathrm{Bd}\, D_{1i} \subset \mathrm{Int}\, E_{1j}\}\}$. By the Side Approximation Theorem there exists a polyhedral 2-sphere S_{22} such that (1) S_{22} is homeomorphically within e_{21} of S, (2) there exist disjoint disks $D_{21}, \cdots,$ and $D_{2n(2)}$ on S, each of diameter less than e_{21}, such that $\mathrm{Cl}(S - \bigcup_{i=1}^{n(2)} D_{2i}) \subset \mathrm{Ext}\, S_{22}$, and (3) there exists disjoint disks $K_{21}, \cdots,$ and $K_{2m(2)}$ on S_{22}, each of diameter less than e_{21}, such that $\mathrm{Cl}(S_{22} - \bigcup_{i=1}^{m(2)} K_{2i}) \subset \mathrm{Int}\, S$. Let e_{22} denote

$$(1/3)\min\{\ \min\{d(D_{2i}, D_{2j}): i \neq j,\ 1 \leq i \leq n(2),\ \text{and}\ 1 \leq j \leq n(2)\},$$

$$\min\{d(K_{2i}, K_{2j}):\ i \neq j,\ 1 \leq i \leq m(2),\text{and}\ 1 \leq j \leq m(2)\},$$

$$\min\{d(D_{2i}, S_{22} - \textstyle\bigcup_{j=1}^{m(2)} K_{2j}):\ 1 \leq i \leq n(2)\}\}.$$

If $M'_{22} = \{z: z \in S_2 \cup \mathrm{Int}\, S_2$ and $d(z, \mathrm{Ext}\, S) \leq e_{22}\}$ then in the same way as M_{11} was obtained, a polyhedral 3-manifold-with-boundary M_{22} exists such that $[S, S_2] \subset \mathrm{Int}\, M_{22}$ and $M_{22} \subset M'_{22}$. It can be assumed that M_{22} and S_{22} are in relative general position. Let M_2 denote the union of M_{22} and $[S_{22}, S_2]$. Now $M_2 - [S_{22}, S_2]$ has a finite number of components, and let $F_{21}, \cdots,$ and $F_{2r(2)}$ be the components of $\mathrm{Cl}(M_2 - [S_{22}, S_2])$. It follows that for each i, the diameter of F_{2i} is at most $5e_2$. It follows from the definition of M_2 that $S \subset \mathrm{Int}\, M_2 \subset M_2 \subset \mathrm{Int}\, M_1$ and that $\{x: x \in C$ and $d(x, S) > e_{22}\}$ is disjoint from $\mathrm{Int}\, M_2$.

There exists an A-subdivision $\{E_{21}, \cdots, E_{2q(2)}\}$ of $[S_{22}, S_2]$ such that (1) for each i, $1 \leq i \leq n(2)$, there exists a positive integer j such that $\mathrm{Bd}\, D_{2i} \subset \mathrm{Int}\, E_{2j}$, (2) for each i, $1 \leq i \leq q(2)$, the diameter of E_{2i} is less than $5e_2$, and (3) $q(2) \geq 4q(1)$. Let A^*_{2i} and A_{2i} be defined as follows: $A^*_{2i} = \bigcup\{X: X \in \{E_{21}, \cdots, E_{2q(2)}\}$ or $X \in \{F_{21}, \cdots, F_{2r(2)}\}$ and $d(X, E_{2i}) < e_2\}$ and A_{2i} is the component of A^*_{2i} that contains E_{2i}. For each p in S, there exists an integer j such that p is in A_{2j}. For each p in S, denote $\bigcup\{A_{2i}: p \in A_{2i}\}$ by A_{p2}. Note that for each p in S, the diameter of A_{p2} is at most $34e_2$. It follows that for each p in S, $A_{p2} \subset \mathrm{Int}\, A_{p1}$. Furthermore, if $p \in [S_{11}, S_1]$ and Z_{p1} is the component of $A_{p1} \cap [S_{11}, S_1]$ that contains p, then $[S_{22}, S_2] \cap A_{p2}$ is contained in Z_{p1} and for each p in S, there exists a 3-cell C_{p2} such that $A_{p2} \subset \mathrm{Int}\, C_{p2} \subset C_{p2} \subset \mathrm{Int}\, A_{p1}$.

Now $S_2 \cap \{\bigcup_{i=1}^{q(1)} \mathrm{Bd}\, E_{1i}\}$ exists and has one and only one component with diameter greater than $1/2$ the diameter of C. Denote this component by W_1. It follows that for each i, $1 \leq i \leq q(1)$, the intersection of W_1 and E_{1i} exists.

There exists a homeomorphism h_2 from $S_2 \cup \mathrm{Int}\, S_2$ onto $B(1.01)$ with the following properties: (1) $h_2[[S_{22}, S_2]] = [\mathrm{Bd}\, B(.99), \mathrm{Bd}\, B(1.01)]$, (2) $h_2[C]$ is contained in $\mathrm{Int}\, B(1.001)$, (3) if $x \in W_1$ then $h_2(x)$ is equal to the radial projection of $h_1(x)$ onto $\mathrm{Bd}\, B(1.01)$, (4) if x is in $C - M_2$, then $h_2(x) = h_1(x)$, (5) h_2 differs from h_1 only at points close to $[S_{22}, S_2]$, and (6) $\{h_2[E_{21}], \cdots, h_2[E_{2q(2)}]\}$ is a canonical subdivision of $[\mathrm{Bd}\, B(.99), \mathrm{Bd}\, B(1.01)]$.

The process is continued by induction. Hence for each positive i, e_i, e_{i1}, e_{i2}, $e_{(i-1)3}$, S_i, S_{ii}, $D_{i1}, \cdots, D_{in(i)}$, $K_{i1}, \cdots, K_{im(i)}$, M_{ii}, M_i, $F_{i1}, \cdots, F_{ir(i)}$, $E_{i1}, \cdots, E_{iq(i)}$, $A_{i1}^*, \cdots, A_{iq(i)}^*$, $A_{i1}, \cdots, A_{iq(i)}$, A_{pi}, $Z_{p(i-1)}$, C_{pi}, W_i, and h_i are defined inductively as follows: (1) Since S is contained in $\operatorname{Int} M_{i-1}$, there exists a positive number $e_{(i-1)3}$ such that if Q is a set of diameter less than $34e_{(i-1)3}$ and the intersection of Q with S exists, then there exists a 3-cell R such that $Q \subset \operatorname{Int} R \subset R \subset \operatorname{Int} M_{i-1}$. (2) Let e_i be $(1/3)\min\{e_{(i-1)3}, 2^{-i}, d(C, \operatorname{Bd} M_{i-1})\}$. (3) S_i is a polyhedral 2-sphere in $\operatorname{Ext} S$ such that S_i is homeomorphically within e_i of S. (4) Let e_{i1} be $(1/3)\min\{d(C, S_i), \min\{d(\operatorname{Bd} D_{(i-1)j}, E_{(i-1)k}) : \operatorname{Bd} D_{(i-1)j} \subset \operatorname{Int} E_{(i-1)k}\}\}$. (5) S_{ii} is a polyhedral 2-sphere that is homeomorphically within e_{i1} of S, such that there exist disjoint disks $D_{i1}, \cdots$, and $D_{in(i)}$ on S, each of diameter less than e_{i1}, with the property that $\operatorname{Cl}(S - \bigcup_{j=1}^{n(i)} D_{ij})$ is contained in $\operatorname{Ext} S_{ii}$, and there exist disjoint disks $K_{i1}, \cdots$, and $K_{im(i)}$ on S_{ii}, each of diameter less than e_{i1}, with the property that $\operatorname{Cl}(S_{ii} - \bigcup_{j=1}^{m(i)} K_{ij})$ is contained in $\operatorname{Int} S$. Such a 2-sphere exists by Bing's Side Approximation Theorem. (6) Let e_{i2} be

$$(1/3) \ \min\{\min\{d(D_{iu}, D_{iv}) : u \neq v, \ 1 \le u \le n(i), \text{ and } 1 \le v \le n(i)\},$$

$$\min\{d(K_{iu}, K_{iv}) : u \neq v, \ 1 \le u \le m(i), \text{ and } 1 \le v \le m(i)\},$$

$$\min\{d(D_{iu}, S_{ii} - \bigcup_{j=1}^{m(i)} K_{ij}) : 1 \le u \le n(i)\}\}.$$

(7) Let M_{ii} be $\{x : x \in S_i \cup \operatorname{Int} S_i \text{ and } d(x, \operatorname{Ext} S) \le e_{i2}\}$ and let M_{ii} be a polyhedral 3-manifold-with-boundary such that $[S, S_i] \subset \operatorname{Int} M_{ii}$ and $M_{ii} \subset M_{ii}'$. It is assumed that M_{ii} and S_{ii} are in relative general position. (8) Let M_i be the union of M_{ii} and $[S_{ii}, S_i]$. (9) The closure of $M_i - [S_{ii}, S_i]$ has a finite number of components, and let $F_{i1}, \cdots$, and $F_{ir(i)}$ be the components of $\operatorname{Cl}(M_i - [S_{ii}, S_i])$. (10) $\{E_{i1}, \cdots, E_{iq(i)}\}$ is an A-subdivision of $[S_{ii}, S_i]$ such that (a) for each j, $1 \le j \le n(i)$, there exists a k such that $\operatorname{Bd} D_{ij} \subset \operatorname{Int} E_{ik}$, (b) for each j, $1 \le j \le q(i)$, the diameter of E_{ij} is less than $5e_i$, and (c) $q(i) \ge q(i-1)$. (11) For each j, $1 \le j \le q(i)$, A_{ij}^* and A_{ij} are defined as follows: $A_{ij}^* = \bigcup\{x : x \in \{E_{i1}, \cdots, E_{iq(i)}\} \text{ or } x \in \{F_{i1}, \cdots, F_{ir(i)}\} \text{ and } d(x, E_{ij}) < e_i\}$, and A_{ij} is the component of A_{ij}^* that contains E_{ij}. (12) For each p in S, let A_{pi} be $\bigcup\{A_{ij} : p \in A_{ij}\}$. (13) For each p in S, if p is in $[S_{ii}, S_i]$ then $Z_{p(i-1)}$ is the component of $A_{pi} \cap [S_{ii}, S_i]$ that contains p. (14) For each p in S, C_{pi} is a 3-cell that contains A_{pi} in its interior and is contained in the interior of M_{i-1}. (15) W_{i-1} is the component of $S_i \cap \{\bigcup_{j=1}^{q(i-1)} \operatorname{Bd} E_{(i-1)j}\}$ with diameter greater than $(1/2) \operatorname{diam} C$. (16) h_i is a homeomorphism from $S_i \cup \operatorname{Int} S_i$ onto $B(1 + 10^{-i})$ such that (a) $h_i[[S_{ii}, S_i]] = [\operatorname{Bd} B(1 - 10^{-i}), \operatorname{Bd} B(1 + 10^{-i})]$, (b) $h_i[C] \subset \operatorname{Int} B(1 - 10^{-i-1})$, (c) if $x \in W_{i-1}$ then $h_i(x)$ is equal to the radial projection of $h_{i-1}(x)$ onto $\operatorname{Bd} B(1 + 10^{-i})$, (d) if x is in $C - M_i$ then $h_i(x) = h_{i-1}(x)$, (e) h_i differs from h_{i-1} only at points close to $[S_{ii}, S_i]$, and (f) $\{h_i[E_{i1}], \cdots, h_i[E_{iq(i)}]\}$ is a canonical subdivision of $[\operatorname{Bd} B(1 - 10^{-i}), \operatorname{Bd} B(1 + 10^{-i})]$.

It follows that $B(1) = \bigcap_{i=1}^{\infty} h_i[S_i \cup \operatorname{Int} S_i]$. Let G be the following decomposition of $B(1)$: $G = \{g : \text{ either (1) for some point } p \text{ of } B(1) \text{ there exists a positive}$

integer n such that $p \notin h_n[M_{ii}]$ and $g = \{p\}$, or (2) there exists a q in S such that $g = \bigcap_{i=1}^{\infty} h_i[A_{qi}]\}$. Then G is an upper semi-continuous decomposition of $B(1)$. It follows from the way the A_{pi}'s and the h_i's are defined that the intersection of each nondegenerate element of G with the boundary of $B(1)$ is a one-point set.

If g is a nondegenerate element of G, then there exists a q in S such that $g = \bigcap_{i=1}^{\infty} h_i[A_{qi}]$. But $\bigcap_{i=1}^{\infty} h_i[A_{qi}]$ is equal to $\bigcap_{i=1}^{\infty} h_i[C_{qi}]$. Hence g is point-like.

Define a function h from C into $B(1)/G$ as follows: If $x \in \text{Int } C$, $h(x) = \{\lim h_i(x)\}$ and if $x \in S$, $h(x) = \bigcap_{i=1}^{\infty} h_i[A_{xi}]$. Then h is one-to-one. If not, there exist points p and q in C such that $p \neq q$ and $h(p) = h(q)$. Clearly, neither p nor q can be in Int C. If p and q are in S then there exists an integer n such that A_{pn} and A_{qn} are disjoint, and hence $h_n[A_{pn}]$ and $h_n[A_{qn}]$ are disjoint. This is a contradiction. Clearly h is onto $B(1)/G$ and h is a homeomorphism on Int C. To show h is a homeomorphism it suffices to show h is continuous at each point of S. Suppose p is in S, $\{a_i\}$ is a sequence in C, and $\{a_i\}$ converges to p. If U is an open set in $B(1)/G$ that contains $h(p)$, then there exists a positive integer k such that $h_k[A_{pk}]$ is contained in U. Since $\{a_i\}$ converges to p, all but finitely many of the a_i's belong to A_{pi}. Therefore all but finitely many of the $h_i(a_i)$'s belong to $h_i[A_{pi}]$. By the way the h_i's are constructed, if $j \geq i$ and $h_i(a_i) \in h_i[A_{pi}]$, then $h_j(a_i) \in h_i[A_{pi}]$, and hence all but finitely many of the $h(a_i)$'s belong to U. Therefore h is continuous at p and h is homeomorphism.

COROLLARY 1. *If C is a crumpled cube then there exists an upper semi-continuous decomposition G of $B(1)$ such that the intersection of each nondegenerate element of G with the boundary of $B(1)$ in a one-point set, each nondegenerate element of G is point-like, and $B(1)/G$ is homeomorphic to C.*

Proof. This follows directly from Theorem 2 and Theorem 3.

COROLLARY 2. *If C and D are crumpled cubes, S is a tame 2-sphere in S^3, and C and D are sewn together along their boundaries, then there exists an upper semi-continuous decomposition G of S^3 such that the intersection of each non-degenerate element of G with S is a one-point set and S^3/G is homeomorphic to the sum of C and D.*

4. Two examples.

EXAMPLE 1. In [3] Bing constructed a 2-sphere S with the property that S is wild at each point and one of the complementary domains of $S^3 - S$ is an open 3-cell. In this example, it is shown that if C is the union of S and the "bad" complementary domain of $S^3 - S$ then the union of two copies of C sewn together along their boundaries by the identity homeomorphism is topologically S^3.

The notation and terminology used will be the same as that used by Bing in [3]. Let X denote the sum of two copies of C sewn together along their boundaries by the identity homeomorphism. An upper semi-continuous decomposition G of

S^3 will be described. It will then be shown that X is homeomorphic to S^3/G and S^3/G is homeomorphic to S^3.

A decomposition of S^3. Let S' denote $\{(x, y, z) : x = 0\} \cup \{\infty\}$, and let f be the homeomorphism defined by $f(x, y, z) = (-x, y, z)$ if $(x, y, z) \in E^3$ and $f(\infty) = \infty$. Let f_1 be a homeomorphism from S^3 onto S^3 such that $f_1[S_0] = S'$ and for each i, $1 \leq i \leq n$, let A_i denote $f_1[T_i] \cup ff_1[T_i]$. Let f_2 be a homeomorphism from S^3 onto S^3 such that $f_2 f_1[S_1] = S'$ and if $x \in S^3 - \bigcup_{i=1}^n A_i$ then $f_2(x) = x$, and for each i and j, $1 \leq i \leq n$, $1 \leq j \leq 15$, let A_{ij} denote $f_2[T_{ij}] \cup ff_2[T_{ij}]$. The process is continued by induction. Let A_0 be $(\bigcup A_i) \cap (\bigcup A_{ij}) \cap (\bigcup A_{ijk}) \cap \cdots$. It is assumed that the f_i's are defined so that each component of A_0 is a tame arc that intersects S' in a one-point set.

Let G denote $\{g : g$ is a component of A_0 or there exists a p in $S^3 - A_0$ such that $\{p\} = g\}$. Then G is an upper semi-continuous decomposition of S^3 and S^3/G is homeomorphic to X.

To prove that S^3/G is homeomorphic to S^3 it will be assumed that S^3 is $\{(x, y, z, w) : |x| + |y| + |z| + |w| = 1\}$ and S' is $\{(x, y, z, w) : (x, y, z, w) \in S^3$ and $w = 0\}$.

If B is the homeomorphic image of A_i then B_{ij} will denote the homeomorphic image of A_{ij}, B_{ijk} denotes the homeomorphic image of A_{ijk}, etc.

LEMMA 6. *If B is the homeomorphic image of A_i in S^3, $F_1, \cdots, F_m$ are polyhedral disks such that (1) for each i, $1 \leq i \leq m$, if C is a component of $F_i \cap B$ then C is a polyhedral disk*, Int $C \subset$ Int B, *and $B - C$ has exactly two components, the closure of each being a solid torus, and (2) for each pair of integers i and j, $1 \leq i \leq m$, $1 \leq j \leq m$, and $i \neq j$, some component of $B - F_i$ contains $B \cap F_j$ and some component of $B - F_j$ contains $B \cap F_i$. Then there is a homeomorphism h and a positive integer j such that h is from S^3 onto S^3, if $x \in S^3 - B$ then $h(x) = x$, and for each $B_{i_1 \ldots i_j}$, $h[B_{i_1 \ldots i_j}]$ intersects at most one of the elements of $\{F_1, \cdots, F_m\}$.*

Proof. Let $Q(m)$ be the statement that Lemma 6 is true if there are m elements in $\{F_1, \cdots, F_m\}$.

If h is the identity homeomorphism and $j = 1$ then $Q(1)$ is true.

Assume $Q(k)$ is true for $k \leq m - 1$. Suppose $F_1, \cdots, F_m$ are polyhedral disks that satisfy the hypotheses of Lemma 6. Let $\{C_1, \cdots, C_k\}$ be the set of components of $(\bigcup F_i) \cap B$ and assume the numbering is such that for each j, $2 \leq j \leq k - 1$, $\bigcup_{i<j} C_i$ and $\cup_{i>j} C_i$ are in different components of $B - C_j$. Then there exists integers r, s, t and v such that $\bigcup_{i=1}^r C_i = F_s \cap B$ and $\bigcup_{i=t}^m C_i = F_v \cap B$. Then there exist a homeomorphism h' such that (1) h' is from S^3 onto S^3, (2) if $x \in S^3 - B$ then $h'(x) = x$, (3) for each i and j, $1 \leq j \leq 15$, and $1 \leq i \leq m$, if C is a component of $h'[B_j] \cap F_i$, then C is a polyhedral disk, Int $C \subset$ Int B, and $h'[B_j] - C$ has exactly two components, the closure of each being a solid torus, (4) for each triple

of integers i, j, k, $1 \leq i \leq m$, $1 \leq j \leq m$, $1 \leq k \leq 15$, and $i \neq j$, such that F_i and F_j intersect $h'[B_k]$, some component of $h'[B_k] - F_j$ contains $F_i \cap h'[B_k]$ and some component of $h'[B_k] - F_i$ contains $F_j \cap h'[B_k]$, and (5) for each k, $1 \leq k \leq 15$, $h'[B_k]$ intersects at most $m - 1$ of the elements of $\{F_1, \cdots, F_m\}$. Hence for each k, $1 \leq k \leq 15$, there exists an integer $z \leq m - 1$ such that $h'[B_k]$ satisfies the hypotheses of Lemma 6 and $h'[B_k]$ intersects at most z of the elements in $\{F_1, \cdots, F_m\}$. Therefore there exists a homeomorphism h_k and an integer $j(k)$ such that h_k is from S^3 onto S^3, if $x \in S^3 - h'[B_k]$ then $h_k(x) = x$, and for each $h'[B_{i_1 \cdots i_{j(k)}}]$, $h_k[h'[B_{i_1 \cdots i_{j(k)}}]]$ intersects at most one of the elements in $\{F_1, \cdots, F_m\}$. Let $h = h_1 h_2 \cdots h_{15} h'$ and let $j = \max\{j(k) : 1 \leq k \leq 15\}$. Then h and j satisfy the conclusion of Lemma 6. Hence $Q(m)$ is true, and the Lemma is proved.

LEMMA 7. *If B is the homeomorphic image of A_i in S^3 and $P_1, \cdots,$ and P_m are disjoint polyhedral 2-spheres in S^3, then there is a homeomorphism h and a positive integer q such that h is from S^3 onto S^3, if $x \in S^3 - B$ then $h(x) = x$, and for each $B_{i_1 \cdots i_q}$, $h[B_{i_1 \cdots i_q}]$ intersects at most one of the elements in $\{P_1, \cdots, P_m\}$.*

Proof. There exists a homeomorphism k from S^3 onto S^3 such that if $x \in S^3 - B$ then $k(x) = x$, for each i and j, $1 \leq j \leq 15$, and $1 \leq i \leq n$, $k[B_{ij}]$ is polyhedral and $k[B_{ij}]$, $P_1, \cdots,$ and P_m are in relative general position, and for each i and j, $1 \leq j \leq 15$, and $1 \leq i \leq n$, if C is a component of $k[B_{ij}] \cap (\bigcup_{i=1}^{m} P_i)$, then C is a polyhedral disk, Int $C \subset$ Int $k[B_{ij}]$, and $k[B_{ij}] - C$ has exactly two components, the closure of each being a solid torus. Let $\{F_1, \cdots, F_r\}$ be the collection of components of $(\bigcup_{j=1}^{15} k[B_{ij}]) \cap (\bigcup_{i=1}^{m} P_i)$.

Lemma 7 then follows from Lemma 6.

Suppose the 2-spheres $P_1, \cdots,$ and P_m in the hypothesis of Lemma 7 satisfy the following conditions: (1) For each r, $1 \leq r \leq m$, there exists a real number $t(r)$ such that $P_r = \{(x, y, z, w) : (x, y, z, w) \in S^3 \text{ and } w = t(r)\}$ and (2) For each r, $2 \leq r \leq m$, $t(r) > t(r-1)$. If v is a real number such that for each r, $2 \leq r \leq m$, $t(r) - t(r-1) < v$, then the homeomorphism h in Lemma 7 can be constructed such that if M is a subset of $h[B_{i_1 \cdots i_q}]$, then the diameter of M is less than $3v$. Therefore Lemma 7 can be strengthened to obtain the following result.

LEMMA 8. *If B is the homeomorphic image of A_i in S^3 and e is a positive number, then there exists a homeomorphism h and a positive integer q such that h is from S^3 onto S^3, if $x \in S^3 - B$ then $h(x) = x$, and for each $B_{i_1 \cdots i_q}$, the diameter of $h[B_{i_1 \cdots i_q}]$ is less than e.*

THEOREM 4. *If G is the upper semi-continuous decomposition of S^3 whose non-degenerate elements are the components of A_0, then S^3 / G is homeomorphic to S^3.*

Proof. It follows from Lemma 8 that for each positive integer j there exists a homeomorphism t_j and an integer $q(j)$ such that t_j leaves $S^3 - \bigcup A_{i_1 \cdots i_j}$ pointwise

fixed, and for each nondegenerate element g of G the diameter of $t_j[g]$ is at most $1/j$. It follows that S^3 is homeomorphic to S^3/G.

EXAMPLE 2. A "nice" crumpled cube D with the property that the union of two copies of D sewn together along their boundaries by the identity homeomorphism is not homeomorphic to S^3.

Description. Let P be the 2-sphere $\{(x,y,z) \in E^3 : x = 0\} \cup \{\infty\}$ and let K be the 3-cell $\{(x,y,z) \in E^3 : x \geq 0\} \cup \{\infty\}$. Let J be the Fox-Artin arc of Figure 6, contained in Int K except for its left endpoint which lies on P. Let G'

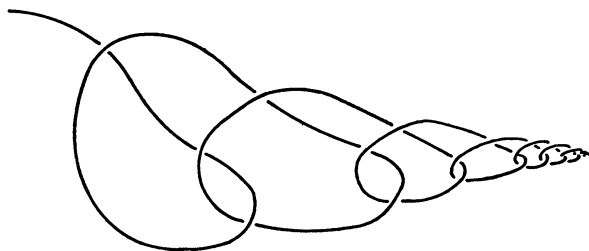FIGURE 6

be the upper semi-continuous decomposition of S^3 whose only nondegenerate element is J. Since J is cellular, S^3/G' is homeomorphic to S^3 and the projection map ϕ from S^3 onto S^3/G' is a homeomorphism on P. Hence $\phi[K]$ is a crumpled cube.

Let G be the upper semi-continuous decomposition of S^3 whose only non-degenerate element is $\bigcup\{(x,y,z) : (|x|,y,z) \in J\}$. Then S^3/G is homeomorphic to the union of two copies of $\phi[K]$ sewn together along their boundaries by the identity homeomorphism. It follows from Example 1.3 of [8] that S^3/G is not homeomorphic to S^3.

BIBLIOGRAPHY

1. B. J. Ball, *The sum of two solid horned spheres*, Ann. of Math. (2) **69** (1959), 253–257.

2. R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. **56** (1952), 354–362.

3. ———, *A wild surface each of whose arcs is tame*, Duke, Math. J. **28** (1961), 1–15.

4. ———, *Approximating surfaces from the side*, Ann. of Math. (2) **77** (1963), 145–192.

5. ———, *Conditions under which a surface in E^3 is tame*, Fund. Math. **47** (1959), 105–139.

6. ———, Mimeographed Notes, Topology Institute, University of Georgia, Summer, 1961.

7. B. G. Casler, *On the sum of two solid Alexander horned spheres*, Notices Amer. Math. Soc. **9** (1962), 108.

8. R. H. Fox and E. Artin, *Some wild cells and spheres in three dimensional space*, Ann. of Math. (2) **49** (1948), 979–990.

9. Norman Hosay, *The sum of a real cube and a crumpled cube is S^3*, Notices Amer. Math. Soc. **10** (1963), 666.

UNIVERSITY OF IOWA,
    IOWA CITY, IOWA